Solutions Exam - Statistics 2019/2020

SOLUTION 1: Define $\bar{X} = \sum_{i=1}^{n} X_i$ and $\overline{X^2} = \sum_{i=1}^{n} X_i^2$.

Then set (I): $\bar{X} = E[X] = \alpha/\beta$ and (II): $Var(X) = \overline{X^2} - \bar{X}^2 = \alpha/\beta^2$.

Solve this system for α and β:
Plugging (I) $\alpha = \bar{X}\beta$ into (II) yields: $\overline{X^2} - \bar{X}^2 = (\bar{X}\beta)/\beta^2 = \bar{X}/\beta$, and so: $\beta = \bar{X}/(\overline{X^2} - \bar{X}^2)$. Plugging this into (I) gives: $\alpha = \bar{X}^2/(\overline{X^2} - \bar{X}^2)$. Thus:

$$\hat{\alpha}_{MOM} = \bar{X}^2/(\overline{X^2} - \bar{X}^2) \qquad \hat{\beta}_{MOM} = \bar{X}/(\overline{X^2} - \bar{X}^2)$$

SOLUTION 2: Note: $f_{\theta,a}(x) = \theta \cdot a^\theta \cdot x^{-\theta-1} \cdot I_{\{x \geq a\}}$, where $I_{\{x \geq a\}} = 1$ if $x \geq a$ and $I_{\{x \geq a\}} = 0$ otherwise. Build the likelihood (joint density):

$$\begin{aligned} L_X(\theta, a) &= f_{\theta,a}(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{\theta,a}(x_i) = \theta^n \cdot a^{n\theta} \cdot \left(\prod_{i=1}^{n} x_i\right)^{-\theta-1} \cdot \prod_{i=1}^{n} I_{\{x_i \geq a\}} \\ &= \theta^n \cdot a^{n\theta} \cdot \left(\prod_{i=1}^{n} x_i\right)^{-\theta-1} \cdot I_{\{\min\{x_1,\ldots,x_n\} \geq a\}} \end{aligned}$$

SOLUTION 2(a): The joint density can be factorized into:

$$\begin{aligned} g(x_1, \ldots, x_n) &= I_{\{\min\{x_1,\ldots,x_n\} \geq a\}} \\ h(t(x_1, \ldots, x_n), \theta) &= \theta^n \cdot a^{n\theta} \cdot \left(\prod_{i=1}^{n} x_i\right)^{-\theta-1} \end{aligned}$$

so that $T(X_1, \ldots, X_n) := \prod_{i=1}^{n} X_i$ is sufficient statistic for θ.

SOLUTION 2(b): The joint density can be factorized into:

$$\begin{aligned} g(x_1, \ldots, x_n) &= \theta^n \cdot \left(\prod_{i=1}^{n} x_i\right)^{-\theta-1} \\ h(t(x_1, \ldots, x_n), a) &= a^{n\theta} \cdot I_{\{\min\{x_1,\ldots,x_n\} \geq a\}} \end{aligned}$$

so that $T(X_1, \ldots, X_n) := \min\{X_1, \ldots, X_n\}$ is sufficient statistic for a.

SOLUTION 2(c): Assume $\min\{x_1, \ldots, x_n\} \geq a$, as otherwise $L_X(\theta, a) = 0$ for all X.

$$l_X(\theta, a) = \log(L_X(\theta, a)) = n\log(\theta) + n\theta\log(a) + (-\theta - 1)\log(\sum_{i=1}^{n} x_i)$$

Take the derivative w.r.t. θ and set it to 0: $\frac{d}{d\theta} l_X(\theta, a) = \frac{n}{\theta} + n\log(a) - \log(\sum_{i=1}^{n} x_i) = 0$.
Solving for θ yields the (potential) ML estimator: $\hat{\theta}_{ML} = \frac{n}{\log(\sum_{i=1}^{n} x_i) - n\log(a)}$
This is indeed the MLE (a maximum point), as: $\frac{d^2}{d\theta^2} l_X(\theta, a) = -\frac{n}{\theta^2} < 0$.

SOLUTION 2(d): The factor $a^{n\theta}$ grows monotonically in a. So a should be as large as possible, subject to $a \leq \min\{x_1, \ldots, x_n\}$. So: $\hat{a}_{ML} = \min\{X_1, \ldots, X_n\}$.

SOLUTION 3(a): $E[\widehat{\sigma^2}] = \frac{1}{n}\sum_{i=1}^{n} E[X_i^2] = \frac{1}{n}\sum_{i=1}^{n}\sigma^2 = \frac{1}{n}\cdot n\sigma^2 = \sigma^2$

Note that $E[X_i^2] = \sigma^2$, as $E[X_i] = 0$ implies: $E[X_i^2] - 0^2 = Var(X_i) = \sigma^2$.

SOLUTION 3(b): Build the log-likelihood (for X_1 only):

$$l_{X_1}(\sigma^2) = \log\left(\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sigma}\cdot\exp\{-\frac{1}{2}\cdot\frac{X_1^2}{\sigma^2}\}\right) = \log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\cdot\log(\sigma^2) - \frac{1}{2}\cdot\frac{1}{\sigma^2}\cdot X_1^2$$

Take the first and the second deriavtive w.r.t. σ^2:

$$\begin{aligned}
\frac{d}{d\sigma^2}\, l_{X_1}(\sigma^2) &= -\frac{1}{2}\cdot\frac{1}{\sigma^2} \quad - \frac{1}{2}\cdot\frac{1}{(\sigma^2)^2}\cdot(-1)\cdot X_1^2 \\
&= -\frac{1}{2}\cdot\frac{1}{\sigma^2} \quad + \frac{1}{2}\cdot\frac{1}{(\sigma^2)^2}\cdot X_1^2 \\
\frac{d^2}{d\sigma^2\, d\sigma^2}\, l_{X_1}(\sigma^2) &= -\frac{1}{2}\cdot\frac{1}{(\sigma^2)^2}\cdot(-1) \quad + \frac{1}{2}\cdot\frac{1}{(\sigma^2)^3}\cdot(-2)\cdot X_1^2 \\
&= +\frac{1}{2}\cdot\frac{1}{(\sigma^2)^2} \quad - \frac{1}{(\sigma^2)^3}\cdot X_1^2
\end{aligned}$$

The expected Fisher information is:

$$\begin{aligned}
I(\sigma^2) &= -E\left[\frac{d^2}{d\sigma^2\, d\sigma^2}\, l_{X_1}(\sigma^2)\right] = -E\left[\frac{1}{2}\cdot\frac{1}{(\sigma^2)^2} - \frac{1}{(\sigma^2)^3}\cdot X_1^2\right] \\
&= -\frac{1}{2}\cdot\frac{1}{(\sigma^2)^2} + \frac{1}{(\sigma^2)^3}\cdot E[X_1^2] = -\frac{1}{2}\cdot\frac{1}{(\sigma^2)^2} + \frac{1}{(\sigma^2)^3}\cdot\sigma^2 = \frac{1}{2\sigma^4}
\end{aligned}$$

SOLUTION 3(c): Yes, it attains the Rao Cramer bound.

The Rao-Cramer bound is equal to: $\frac{1}{n\cdot I(\sigma^2)} = \frac{1}{n\cdot\frac{1}{2\sigma^4}} = \frac{2\sigma^4}{n}$

$$\begin{aligned}
Var(\widehat{\sigma^2}) &= \frac{1}{n^2}\cdot Var\left(\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n^2}\cdot Var\left(\sigma^2\cdot\sum_{i=1}^{n}\left(\frac{X_i}{\sigma}\right)^2\right) \\
&= \frac{\sigma^4}{n^2}\cdot Var\left(\sum_{i=1}^{n}\left(\frac{X_i}{\sigma}\right)^2\right) = \frac{\sigma^4}{n^2}\cdot 2n = \frac{2\sigma^4}{n}
\end{aligned}$$

Note that $X_i \sim N(0,\sigma^2)$ implies $\frac{1}{\sigma}X_i \sim N(0,1)$, so that (see hint):
$\left(\sum_{i=1}^{n}\left(\frac{X_i}{\sigma}\right)^2\right)$ is χ_n^2 distributed and has variance $2n$.

SOLUTION 4: Consider the ratio of the joint densities:

$$\begin{aligned}\lambda(X_1,\ldots,X_9) &= \frac{f_{0,1}(X_1,\ldots,X_9)}{f_{1,1}(X_1,\ldots,X_9)} = \frac{\prod_{i=1}^{9} f_{0,1}(X_i)}{\prod_{i=1}^{9} f_{1,1}(X_i)} = \frac{\prod_{i=1}^{9} \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{1}\cdot\exp\{-\frac{1}{2}\frac{(X_i-0)^2}{1^2}\}}{\prod_{i=1}^{9} \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{1}\cdot\exp\{-\frac{1}{2}\frac{(X_i-1)^2}{1^2}\}} \\ &= \frac{\exp\{-\frac{1}{2}\sum_{i=1}^{9} X_i^2\}}{\exp\{-\frac{1}{2}\sum_{i=1}^{9}(X_i-1)^2\}} = \exp\{-\frac{1}{2}\sum_{i=1}^{9} X_i^2 + \frac{1}{2}\sum_{i=1}^{9}(X_i^2-2X_i+1)\} \\ &= \exp\{4.5-\sum_{i=1}^{9} X_i\}\end{aligned}$$

Under H_0 we have: $\sum_{i=1}^{9} X_i \sim N(0,9)$, so that $\frac{1}{3}\sum_{i=1}^{9} X_i \sim N(0,1)$.

$$P_{H_0}(\lambda(X_1,\ldots,X_9)<k) \;\Leftrightarrow\; P_{H_0}(\exp\{4.5-\sum_{i=1}^{9} X_i\}<k)$$

$$\Leftrightarrow P_{H_0}(\sum_{i=1}^{9} X_i > 4.5-\log(k)) \;\Leftrightarrow\; P_{H_0}(\frac{1}{3}\sum_{i=1}^{9} X_i > 1.5-\log(k)/3)$$

So $(1.5-\log(k)/3)$ must be the 0.9-quantile $q_{0.9}=1.3$ of the $N(0,1)$ distribution.

$$1.5-\log(k)/3 = 1.3 \Leftrightarrow k=\exp(0.6)\approx 1.82$$

Hence, the UMP rejects H_0 if: $\lambda(X_1,\ldots,X_9)<1.82$.

Power of the UMP test: Under H_1 we have: $(\frac{1}{3}\sum_{i=1}^{9} X_i - 3) \sim N(0,1)$.

$$P_{H_1}(\lambda(X_1,\ldots,X_9)<k) \Leftrightarrow P_{H_1}(\frac{1}{3}\sum_{i=1}^{9} X_i > 1.3) \Leftrightarrow P_{H_1}(\frac{1}{3}\sum_{i=1}^{9} X_i - 3 > -1.7)$$

As -1.7 is smaller than the $q_{0.05}=-q_{0.95}=-1.6$ quantile of the $N(0,1)$, **the power of the UMP test is greater than** 0.95.

SOLUTION 5(a): Compute the log likelihood:

$$\begin{aligned}l_X(\theta) &= \log\left(\prod_{i=1}^{n}\binom{x_i+r-1}{x_i}\cdot(1-\theta)^r\cdot\theta^{x_i}\right) \\ &= \log\left((\prod_{i=1}^{n}\binom{x_i+r-1}{x_i})\cdot(1-\theta)^{nr}\cdot\theta^{\sum_{i=1}^n x_i}\right) \\ &= \log\left(\prod_{i=1}^{n}\binom{x_i+r-1}{x_i}\right) + nr\log(1-\theta) + (\sum_{i=1}^{n} x_i)\log(\theta)\end{aligned}$$

Take the derivative w.r.t. θ and set it to 0:

$$\frac{-nr}{1-\theta}+\frac{\sum_{i=1}^n x_i}{\theta}=0 \;\Leftrightarrow\; -nr\theta+(\sum_{i=1}^{n} x_i)(1-\theta)=0 \Leftrightarrow -(nr+\sum_{i=1}^{n} x_i)\theta+\sum_{i=1}^{n} x_i=0$$

$$\Leftrightarrow\; \theta=\frac{\sum_{i=1}^n x_i}{nr+\sum_{i=1}^n x_i} \Leftrightarrow \theta=\frac{\bar{x}}{r+\bar{x}}$$

For the second derivative we have:

$$\frac{d^2}{d\theta^2} l_X(\theta) = \frac{-nr}{(1-\theta)^2} - \frac{\sum_{i=1}^n x_i}{\theta^2} < 0$$

This confirms that $\hat{\theta}_{ML} = \bar{X}/(r + \bar{X})$.

SOLUTION 5(b): For $n = 1$ we have the 2nd derivative of the log likelihood:

$$\frac{d^2}{d\theta^2} l_{X_1}(\theta) = \frac{-r}{(1-\theta)^2} - \frac{X_1}{\theta^2}$$

Compute the Fisher Information:

$$\begin{aligned} I(\theta) &= -E_\theta\left[\frac{d^2}{d\theta^2} l_{X_1}(\theta)\right] = E_\theta\left[\frac{r}{(1-\theta)^2} + \frac{X_1}{\theta^2}\right] = \frac{r}{(1-\theta)^2} + \frac{E[X_1]}{\theta^2} \\ &= \frac{r}{(1-\theta)^2} + \frac{\frac{r\theta}{(1-\theta)}}{\theta^2} = \frac{r}{(1-\theta)^2} + \frac{r\theta}{(1-\theta)\theta^2} = \frac{r\theta + r(1-\theta)}{(1-\theta)^2\theta} = \frac{r}{\theta(1-\theta)^2} \end{aligned}$$

SOLUTION 5(c): Asymptotically $\sqrt{I(\theta)}\sqrt{n} \cdot (\hat{\theta}_{ML} - \theta) \sim \mathcal{N}(0, 1)$, hence:

$$\begin{aligned} & P(q_{0.025} \leq \sqrt{I(\theta)}\sqrt{n} \cdot (\hat{\theta}_{ML} - \theta) \leq q_{0.975}) = 0.95 \\ \Leftrightarrow \quad & P(\hat{\theta}_{ML} - \frac{q_{0.975}}{\sqrt{I(\theta)} \cdot \sqrt{n}} \leq \theta \leq \hat{\theta}_{ML} - \frac{q_{0.025}}{\sqrt{I(\theta)} \cdot \sqrt{n}}) = 0.95 \end{aligned}$$

With $q_{0.975} = 2$ and $q_{0.025} = -2$, and $I(\theta)$ being replaced by $I(\hat{\theta}_{ML})$, we get the CI:

$$\hat{\theta}_{ML} \pm 2/(\sqrt{I(\hat{\theta}_{ML})} \cdot \sqrt{n})$$

Here we have $\hat{\theta}_{ML} = 0.8$ and $2/(\sqrt{I(\hat{\theta}_{ML})}\sqrt{n}) = 2/(\sqrt{2/(0.8 \cdot 0.2^2)}\sqrt{20} \approx 0.057$.

So the two-sided CI is: $[0.743, 0.857]$.

SOLUTION 5(d): Like part (c), but here we use:

$$P(q_{0.05} \leq \sqrt{I(\theta)}\sqrt{n} \cdot (\hat{\theta}_{ML} - \theta)) = 0.95 \Leftrightarrow P(\hat{\theta}_{ML} - \frac{q_{0.05}}{\sqrt{I(\theta)} \cdot \sqrt{n}} \geq \theta) = 0.95$$

With $q_{0.05} = -1.6$ the one-sided 95% CI for θ is: $(-\infty, \hat{\theta}_{ML} + \frac{1.6}{\sqrt{I(\hat{\theta}_{ML})} \cdot \sqrt{n}}]$

Here we have $\hat{\theta}_{ML} = 0.8$ and $\frac{1.6}{\sqrt{I(\hat{\theta}_{ML})}\sqrt{n}} = \frac{1.6}{\sqrt{\frac{2}{0.8 \cdot 0.2^2}}\sqrt{20}} \approx 0.045$.

So the one-sided CI is: $(-\infty, 0.845]$.

SOLUTION 5(e): Asymptotically: $\frac{\frac{d}{d\theta} l_X(\theta)}{\sqrt{n \cdot I(\theta)}} \sim N(0, 1)$ where $\frac{d}{d\theta} l_X(\theta) = \frac{-nr}{1-\theta} + \frac{\sum_{i=1}^n x_i}{\theta}$.

Given $r = 2$, $\bar{X} = 8$ and $n = 20$ and $\theta_0 = 0.9$ we get:
$\frac{-nr}{1-\theta} + \frac{\sum_{i=1}^n x_i}{\theta} = \frac{-40}{1-0.9} + \frac{20 \cdot 8}{0.9} \approx -222$ and $\sqrt{n \cdot I(\theta)} = \sqrt{20 \cdot \frac{2}{0.9 \cdot 0.1^2}} \approx 66.67$

Therefore the score test statistic takes the value: $\frac{\frac{d}{d\theta} l_X(\theta)}{\sqrt{n \cdot I(\theta)}} = \frac{-222}{66.67} \approx -3.33$. As the value is lower than the $q_{0.01}$ quantile -2.3 of the $N(0, 1)$, **the score test would reject the null hypothesis** to the level 0.02.